On Hide's Magnetic Analogue of Ertel's Vorticity Theorem

W. Schröder[a] and H.-J. Treder[b]

[a] Hechelstr. 8, D-28777 Bremen-Roennenbeck
[b] Rosa-Luxemburg-Str. 17a, D-14482 Potsdam

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The relativistic formulation of Hide's "magnetic analogue" of Ertel's potential vorticity theorem is Dirac's "new classical theory of electrons".

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Dirac's "new classical theory of electrons" [1, 2] describes the relativistic hydrodynamics of conductive media with an "Ohm law"

$$u^\mu = \frac{e}{mc}(A^\mu - g^{\mu\nu}\partial_\nu S), \quad (\mu, \nu = 0, 1, 2, 3). \quad (1)$$

In (1), $u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}t}$ is the four-dimensional velocity, $\mathrm{d}\tau$ the differential of the proper time, A^μ the four-dimensional electro-magnetic vector potential, and S a gauge function. $g_{\mu\nu}$ means the metrical Minkowski tensor, and e and m are the charge and the rest mass of an electron, respectively.

According to (1), the norm of the velocity is

$$-c^2 = u_\mu u^\mu = \frac{e^2}{m^2c^2}(A_\mu A^\mu + g^{\mu\nu}\partial_\mu S\,\partial_\nu S - 2A^\mu\partial_\mu S). \quad (2)$$

The tensor of the four-dimensional rotation is

$$\omega_{\mu\nu} = \partial_\nu u_\mu - \partial_\mu u_\nu = \frac{e}{mc}(\partial_\nu A_\mu - \partial_\mu A_\nu) = \frac{e}{mc}F_{\mu\nu} = -\omega_{\nu\mu}, \quad (3)$$

where $F_{\mu\nu}$ is the anti-symmetric Maxwellian tensor of the electro-magnetic field.

The four-dimensional relativistic generalization of Helmholtz' theorem of the conservation of vorticity means [3–5]

$$\mathrm{d}\omega_{\alpha\beta} = u^\lambda\partial_\lambda\omega_{\alpha\beta} = \omega^\lambda{}_\alpha\partial_\beta u_\lambda - \omega^\lambda{}_\beta\partial_\alpha u_\lambda + \partial_\beta\frac{\mathrm{d}u_\alpha}{\mathrm{d}\tau} - \partial_\alpha\frac{\mathrm{d}u_\beta}{\mathrm{d}\tau}, \quad (4)$$

where $\frac{\mathrm{d}\Phi}{\mathrm{d}\tau} = u^\lambda\partial_\lambda\Phi$.

Because of the identity $\frac{\mathrm{d}}{\mathrm{d}\tau}\mathrm{d}x^\alpha = u^\lambda\partial_\lambda\mathrm{d}x^\alpha$ one can write (4), with the help of the Cartan exterior differential [6]

$$[\mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta] = -[\mathrm{d}x^\beta \wedge \mathrm{d}x^\alpha],$$

like a relativistic formulation of Ertel's vorticity theorem [7]; see also [8]

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(\omega_{\mu\nu}[\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu]) = 2\partial_\nu\frac{\mathrm{d}u_\mu}{\mathrm{d}\tau}[\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu]. \quad (5)$$

By the Cartan-Stokes integral theorem, (5) yields the circulation theorem [6]

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{C_2}\omega_{\mu\nu}[\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu] = 2\oint_{C_1}\frac{\mathrm{d}u_\mu}{\mathrm{d}\tau}\mathrm{d}x^\mu \quad (6)$$

(with $C_1 = \partial C_2$).

According to Poincaré's rules of exterior differentiation, (5, 6) with Dirac's law (1) yield an analogous equation for the electromagnetic field strength $F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu$, namely the non-holonomic expression

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(F_{\mu\nu}[A^\mu \wedge A^\nu]) = 2\partial_\nu\frac{\mathrm{d}A_\mu}{\mathrm{d}\tau}[A^\mu \wedge A^\nu], \quad (7)$$

together with a circulation theorem of the electromagnetic field

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{C_2}F_{\mu\nu}[\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu] = 2\oint_{C_1}\frac{\mathrm{d}A_\mu}{\mathrm{d}\tau}\mathrm{d}x^\mu \quad (8)$$

according to the holonomic equations

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(F_{\mu\nu}[\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu]) = 2\partial_\nu\frac{\mathrm{d}A_\mu}{\mathrm{d}\tau}[\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu]. \quad (9)$$

These equations are a relativistic generalization of Hide's magnetic analogue [9, 10] of Ertel's vorticity theorem. The relativistic point of view generalizes the "magnetic analogue" of Hide to an "electro-magnetic" analogue.

Reprint requests to W. Schröder.

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